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On functional analysis of n-normed spaces

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Chapter 1

Preliminaries

1.1 Abstract

In this essay, we prove fixed point theorems for $\mathfrak{n}$-normed spaces which are already known in normed spaces (Edelstein's, Weissinger's and Banach's). Then we present the generalization of the linear and non-linear Lax-Milgram theorems, utilizing a general version Zarantonello theorem in the process.

1.2 Introduction

In the field of functional analysis, particularly useful results are fixed point theorems, as they have proven to be helpful in finding unique solutions to algebraic equations, differential equations and integral equations and as a tool for proofs even in other areas of mathematics. On inner product spaces, the study of bounded linear functionals, dual spaces and their connection has also been found beneficial.

In 1964 S. Gähler introduced the notion of linear 2-normed spaces ([3]). Later on the concept has been generalized and mathematicians began studying $\mathfrak{n}$-normed spaces and generalizing notions such as convergence, completeness and their topologies. In particular, T. K. Samantha and coauthors have studied the Uniform Boundedness Principle and Hahn-Banach Theorem in linear $\mathfrak{n}$-normed spaces, which allowed the study of reflexivity in linear $\mathfrak{n}$-normed spaces ([7]). J. Brzdek and K. Cieplinski have examined fixed point theorems in $\mathfrak{n}$-Banach spaces from the perspective of Ulam-stability of eigenvectors and they have applied them to functional and differential equations ([1]). Furthermore, the theory of $\mathfrak{n}$-Hilbert spaces have been developed (first for the $\mathfrak{n} = 2$ case in the 1973, by C. Diminnie, S. Gähler and A. White ([2]), then in 1989 A. Misiak for $\mathfrak{n} \geq 2$ ([12])), resulting in the generalization of the Schwarz-inequality, polarization identities and the Riesz representation theorem which connects the theory of bounded linear functionals to dual spaces.

In a lot of cases, the functional analysis in $\mathfrak{n}$-normed spaces is only possible, or can be more generically studied through fixing all but one variables of the $\mathfrak{n}$-norm and defining $\mathfrak{b}$-convergence and $\mathfrak{b}$-sesquilinearity.

The aim of this essay is to discuss some fixed point theorems in $\mathfrak{n}$-Banach spaces and to show how some of the highly important theorems can be generalized in $\mathfrak{n}$-Hilbert spaces.

We organize our essay as follows. In the next chapter, we summarize the basic notions regarding the $\mathfrak{n}$-norm and their consequences. The chapter that follows is about the generalization of three fixed point theorems such as Edelstein fixed point theorem and the Weissinger fixed point theorem, resp. the Banach fixed point theorem. After that the generalization of some famous theorems such as Riesz reprezentation theorem, Lax-Milgram theorem will be presented and proved.

Budapest, 17 July 2024

András Sike

Chapter 2

Basic notions

In this chapter we list all definitions and theorems which are required to understand the following chapters.

Definition. Let $n \in \mathbb{N}$ an let $\mathbb{V}$ be a linear space with $\dim \mathbb{V} \geq n$ over the field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. A function $\|\cdot,\ldots,\cdot\| : \mathbb{V}^n \to \mathbb{R}$ is called an **n-norm** on $\mathbb{V}$ and the pair $(\mathbb{V}, \|\cdot,\ldots,\cdot\|)$ is called a **linear n-normed space** if the following hold for all $x,y,x_1,\ldots,x_n \in \mathbb{V}$ and for every $\alpha \in \mathbb{K}$:

(N1) $\|x_1,\ldots,x_n\| = 0 \quad \Longleftrightarrow \quad \{x_1,\ldots,x_n\}$ are linearly dependent

(N2) $\|x_1,\ldots,x_n\| = \left\|x_{\phi(1)},\ldots,x_{\phi(n)}\right\|$ for every ϕ permutation of $\{1,\ldots,n\}$;

(N3) $\|\alpha x_1,x_2,\ldots,x_n\| = |\alpha| \, \|x_1,x_2,\ldots,x_n\|$;

(N4) $\|x+y,x_2,\ldots,x_n\| \leq \|x,x_2,\ldots,x_n\| + \|y,x_2,\ldots,x_n\|$ (**triangle inequality**).

The most standard example for a 2-normed space is $\mathbb{R}^2$ equipped with the following two-norm (cf. [13])

$$\|\mathbf{u},\mathbf{v}\| := \left| \det \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \\ \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \right| \qquad (\mathbf{u},\mathbf{v} \in \mathbb{R}^{\not\vDash}).$$

Another important example of a 2-normed linear space is the usual norm applied to the cross product of vectors in $\mathbb{R}^3$:

$$\|\mathbf{u},\mathbf{v}\| := \|\mathbf{u} \times \mathbf{v}\|_2 = \|\mathbf{u}\|_2 \cdot \|\mathbf{v}\|_2 \cdot \sin(\alpha) \qquad (\mathbf{u},\mathbf{v} \in \mathbb{R}^3)$$

where α is the angle between the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

It is well known (cf. e.g. [9], [14]) that the notion of cross product can be generalized for the space $\mathbb{R}^{n+1}$ as follows:

Definition. If $e_1, \dots e_{n+1}$ is a basis in $\mathbb{R}^{n+1}$ then the cross product of the vectors $x_1, \dots, x_n \in \mathbb{R}^{n+1}$ can be defined as follows:

$$x_1 \times \cdots \times x_n := \det \begin{bmatrix} e_1 & \dots & e_{n+1} \\ x_{1,1} & \dots & x_{1,n+1} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,n+1} \end{bmatrix}$$

where $x_{i,j}$ denotes the j-th coordinate of the vector x_i.

We have proven the following theorem.

Theorem. If $e_1, \dots e_{n+1}$ is a basis in $\mathbb{R}^{n+1}$, then

$$\|x_1, \dots, x_n\| := \|x_1 \times \cdots \times x_n\|_2 = \left\| \sum_{i=1}^{n+1} (-1)^{i-1} \det(E_i)\, e_i \right\|_2 = \left(\sum_{i=1}^{n+1} \det(E_i)^2 \right)^{\frac{1}{2}}$$

is an n-norm, where E_i is the matrix

$$\begin{bmatrix} x_{1,1} & \dots & x_{1,n+1} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,n+1} \end{bmatrix}$$

without the i-th column.

Proof. We have to show that the four criteria of the n-norm are fulfilled.

(N2) Swapping rows will only change the determinant by a factor of ± 1, therefore for every i we know that $\det(E_i)^2$ will stay the same, implying that their sum and its' square root will stay the same.

(N3) From **(N2)** it is sufficient to see that the absolute homogeneity holds true for the first row. If E_i^{α} denote the matrices which are E_i, but every $x_{1,j}$ is replaced by $\alpha x_{1,j}$:

$$\begin{aligned} \|\alpha x_1, \dots, x_n\| &= \left(\sum_{i=1}^{n+1} \det(E_i^{\alpha})^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n+1} (\alpha \det(E_i))^2 \right)^{\frac{1}{2}} \\ &= \left(\alpha^2 \sum_{i=1}^{n+1} \det(E_i)^2 \right)^{\frac{1}{2}} = |\alpha| \cdot \|x_1, \dots, x_n\| . \end{aligned}$$

(N4) If $x_i = y + z$, and the respective matrices are E_i, E_i^y, E_i^z, from the determinant's linear property, we have

$$\begin{aligned}
\|x_1,\ldots,x_{i-1},y+z,x_{i+1},\ldots,x_n\| &= \left(\sum_{i=1}^{n+1}\det(E_i)^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n+1}\det(E_i^y+E_i^z)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n+1}\det(E_i^y)^2+\det(E_i^z)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n+1}\det(E_i^y)\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n+1}\det(E_i^z)\right)^{\frac{1}{2}} \\
&= \|x_1,\ldots,x_{i-1},y,x_{i+1}\ldots,x_n\| \\
&\quad + \|x_1,\ldots,x_{i-1},z,x_{i+1}\ldots,x_n\|
\end{aligned}$$

(N1) If $\{x_1,\ldots x_n\}$ are linearly dependent, there exist α_j for which $\sum_{j=2}^{n}\alpha_j x_j = x_1$, and not all $\alpha_j = 0$. From **(N3)** and **(N4)** we get that

$$0 \leq \|x_1,\ldots,x_n\| = \left\|\sum_{j=2}^{n}\alpha_j x_j,x_2,\ldots,x_n\right\| \leq \sum_{j=2}^{n}|\alpha_j|\cdot\|x_j,x_2,\ldots,x_n\|.$$

So it suffices to prove that $\|x_j,x_2,\ldots,x_n\| = 0$ for every j.

$$\|x_j,x_2,\ldots,x_n\| = \left(\sum_{i=1}^{n+1}\det(E_i)^2\right)^{\frac{1}{2}} = 0$$

since every E_i has two identical rows meaning that for every i : $\det(E_i) = 0$. Conversely, suppose $\|x_1,\ldots,x_n\| = 0$ which is equivalent to $\det(E_i) = 0$ for every i. This means that for every i, the vectors $\{\{x_{1,j}\ldots x_{n,j}\} : j \in \{1,\ldots n+1\}\setminus\{i\}\}$ are linearly dependent. Thus, from the $n+1$ vectors any n are linearly dependent, meaning all $n+1$ are. ■

Definition. A sequence $(x_k) \subseteq \mathbb{V}$ is said to converge to $x \in \mathbb{V}$ if for every $e_2, \ldots, e_n \in \mathbb{V}$

$$\lim_{k\to\infty} \|x_k - x, e_2, \ldots, e_n\| = 0$$

and it is called a Cauchy-sequence if for every $e_2, \ldots, e_n \in \mathbb{V}$

$$\lim_{l,k\to\infty} \|x_k - x_l, e_2, \ldots, e_n\| = 0.$$

The space $\mathbb{V}$ is complete, or n-Banach space if every Cauchy sequence is also convergent in $\mathbb{V}$.

Definition.

$$B_{e_2,\ldots,e_n}(a,\delta) = \{x \in \mathbb{V} : \|x - a, e_2, \ldots, e_n\| < \delta\}$$

is called an open ball in $\mathbb{V}$, and

$$\overline{B_{e_2,\ldots,e_n}(a,\delta)} = \{x \in \mathbb{V} : \|x - a, e_2, \ldots, e_n\| \leq \delta\}$$

is called a closed ball in $\mathbb{V}$, where $\delta \in \mathbb{R}^+$ and $a, e_2, \ldots, e_n \in \mathbb{V}$.

Definition. Let $G \subseteq \mathbb{V}$ is said to be open in $\mathbb{V}$ if for all $a \in G$ there exist $e_2, \ldots, e_n \in \mathbb{V}$ and $\delta > 0$ such that $B_{e_2,\ldots,e_n}(a,\delta) \subseteq G$.

Definition. Let $G \subseteq \mathbb{V}$. Then $\bar{G}$ is called the closure of G and is defined as follows:

$$\bar{G} = \{x \in \mathbb{V} : \exists (x_k) \in G, \lim_{k\to\infty} x_k = x\}$$

A set G is called a closed set if $G = \overline{G}$.

Definition. A subset $G \subseteq \mathbb{V}$ is said to be compact if for any $(x_k) \subseteq \mathbb{V}$ sequence there exists an $(x_l) \subseteq (x_k)$ subsequence which is convergent.

Definition. A sequence $(x_k) \subseteq \mathbb{V}$ is said to be a b-Cauchy sequence if for every $\epsilon > 0$ there exists $N > 0$ such that for every $k,l \geq N$, $\|x_k - x_l, b_2, \ldots, b_n\| < \epsilon$.

Definition. A sequence $(x_k) \subseteq \mathbb{V}$ is said to be a b-converging to x, if $\lim_{k\to\infty} \|x_k - x, b_2, \ldots, b_n\| = 0$. The space $\mathbb{V}$ is b-complete if every b-Cauchy sequence is b-convergent.

Throughout the essay, unless otherwise mentioned, we assume $\{b_2, \ldots, b_n\}$ are linearly indepenent.

Definition. Let $n \in \mathbb{N}$ and $\mathbb{H}$ be a linear space with $\dim \mathbb{H} \geq n$ over the field $\mathbb{K}$. A function $\langle \cdot, \cdot \,|\, \cdot, \ldots, \cdot \rangle : \mathbb{H}^{n+1} \to \mathbb{K}$ is called an n-inner product on $\mathbb{H}$ if for all $x, y, z, x_2, \ldots, x_n \in \mathbb{H}$ and for all $\alpha \in \mathbb{K}$ the following hold:

1. $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$ and $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $\{x_1, \ldots, x_n\}$ are linearly dependent
2. $\langle x, y | x_2, \ldots, x_n \rangle = \langle x, y | x_{\phi(2)}, \ldots, x_{\phi(n)} \rangle$ for every ϕ permutation of $2, \ldots, n$
3. $\langle x, y | x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle$
4. $\langle \alpha x, y | x_2, \ldots, x_n \rangle = \alpha \langle x, y | x_2, \ldots, x_n \rangle$
5. $\langle x + y, z | x_2, \ldots, x_n \rangle = \langle x, z | x_2, \ldots, x_n \rangle + \langle y, z | x_2, \ldots, x_n \rangle$

The pair $(\mathbb{H}, \langle \cdot, \cdot \,|\, \cdot, \ldots, \cdot \rangle)$ is called an n-inner product space.

Theorem. Let $\mathbb{H}$ be a n-inner product space. Then

$$\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \ldots, x_n \rangle}$$

defines a n-norm on $\mathbb{H}$.

Definition. The n-inner product space $\mathbb{H}$ is said to be a n-Hilbert space if it is a complete n-normed space with respect to its induce norm.

Definition. Let $\mathbb{V}$ be a subspace of $\mathbb{H}$ and $b_2,\ldots,b_n$ be fixed elements in $\mathbb{H}$. If $\langle b_i\rangle$ denotes the subspaces of $\mathbb{H}$ generated by b_i, then a map $T : \mathbb{V}\times\langle b_2\rangle\times\cdots\times\langle b_n\rangle \to \mathbb{K}$ is called a b-linear functional on $\mathbb{V}\times\langle b_2\rangle\times\cdots\times\langle b_n\rangle$, if for every $x,y\in\mathbb{V}$ and $\alpha\in\mathbb{K}$ the following hold:

- $T(x+y,b_2,\ldots,b_n) = T(x,b_2,\ldots,b_n) + T(y,b_2,\ldots,b_n)$
- $T(\alpha x,b_2,\ldots,b_n) = \alpha T(x,b_2,\ldots,b_n)$

A b-linear functional is said to be b-bounded if for some $M>0$ such that for every $x\in\mathbb{V}$

$$|T(x,b_2,\ldots,b_n)| \le M\,\|x,b_2,\ldots,b_n\|$$

and the infimum of all such M is defined as the norm of the functional:

$$\|T\| = \inf\{M>0 : |T(x,b_2,\ldots,b_n)| \le M\,\|x,b_2,\ldots,b_n\|\ \forall\, x\in\mathbb{V}\}.$$

Definition. Let $\mathbb{H}$ be an n-Hilbert space, $S\subseteq\mathbb{H}$. $x,y\in\mathbb{H}$ are called b-orthogonal (in notation: $x\perp y$) if $\langle x,y|b_2,\ldots,b_n\rangle = 0$ If for every $s\in S$, $x\perp s$ holds, we say that x is b-orthogonal to S, or in notation $x\perp S$. The set of all elements of $\mathbb{H}$ that are b-orthogonal to S is called the b-orthogonal complement of S and is denoted by $S^{\perp}$.

Definition. Let $\mathbb{H}$ be a linear space over field $\mathbb{K}$. A b-sesquilinear functional (or b-sesquilinear form) T is a mapping

$$T : \mathbb{H}\times\mathbb{H}\times\langle b_2\rangle\times\cdots\times\langle b_n\rangle \to \mathbb{K}$$

where for every $x,y,z\in\mathbb{H}$ and for all $\alpha,\beta\in\mathbb{K}$ the following hold true:

1. $T(x+y,z,b_2,\ldots,b_n) = T(x,z,b_2,\ldots,b_n) + T(y,z,b_2,\ldots,b_n)$
2. $T(\alpha x,y,b_2,\ldots,b_n) = \alpha T(x,y,b_2,\ldots,b_n)$
3. $T(x,y+z,b_2,\ldots,b_n)\ \ T(x,y,b_2,\ldots,b_n) + T(x,y,b_2,\ldots,b_n)$
4. $T(x,\beta y,b_2,\ldots,b_n) = \bar{\beta}T(x,y,b_2,\ldots,b_n)$

T is said to be b-bounded if there exists $M>0$ such that for all $x,y\in\mathbb{H}$

$$|T(x,y,b_2,\ldots,b_n)| \le M\,\|x,b_2,\ldots,b_n\|\,\|y,b_2,\ldots,b_n\|$$

and similarly, the infimum of all such M is defined as the norm of T, denoted by $\|T\|$.

Definition. $T : \mathbb{H} \times \mathbb{H} \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{K}$ b-sesquilinear functional is called b-elliptic, if there exists $\alpha > 0$ where for every $x \in \mathbb{H}$ the following holds true:

$$\Re\left(T\left(x,x,b_2,\ldots,b_n\right)\right) \geq \alpha \left\|x,b_2,\ldots,b_n\right\|.$$

Theorem (generalized Riesz-Fréchet theorem). Let $\mathbb{H}$ be a n-Hilbert space. Then T is a bounded b-linear functional on $\mathbb{H} \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ if and only if there exists a unique $z \in \mathbb{H}$ where $\{z,b_2,\ldots,b_n\}$ are linearly independent and for all $x \in \mathbb{H}$:

$$T\left(x,b_2,\ldots,b_n\right) = \langle x,z|b_2,\ldots,b_2\rangle$$

and furthermore, $\|T\| = \|z,b_2,\ldots,b_n\|$.

Definition. Let $\mathbb{H}$ be an n-Hilbert space and $A : \mathbb{H} \to \mathbb{H}$ operator. A^* is called the adjoint operator of A if for every $x_1,x_2,\ldots,x_{n+1}$ the following holds true:

$$\langle Ax_1,x_2|x_3,\ldots,x_{n+1}\rangle = \langle A^*x_1,x_2|x_3,\ldots,x_{n+1}\rangle.$$

Chapter 3

Fixed points in n-Banach spaces

In this chapter we formulate and prove the generalization of some classical fixed point theorems such as

- Edelstein fixed point theorem,
- Weissinger fixed point thorem,
- Banach fixed point theorem.

3.1 The generalized Edelstein fixed point theorem

Theorem. Let $(\mathbb{V}, \|\cdot,\dots,\cdot\|)$ be an n-Banach space, $\mathcal{H} \subset \mathbb{V}$, $\phi : \mathcal{H} \to \mathcal{H}$ mapping such that for some $b_2, \dots b_n \in \mathbb{V}$ and for all $u,v \in \mathcal{H}, u \neq v$:

$$\|\phi(u) - \phi(v), b_2, \dots, b_n\| < \|u - v, b_2, \dots, b_n\|$$

holds. Then ϕ has at most one fixed point in $\mathcal{H}$, that is

$$u,v \in \mathcal{H} : \qquad \phi(u) = \phi(v)$$

implies $u = v$. Furthermore, if $\mathcal{H}$ is compact, then this fixed point is unique.

Proof.

Step 1. Suppose that there are some $u,v \in \mathcal{H}$, $u \neq v$ such that $\phi(u) = u$ and $\phi(v) = v$ hold. Then it follows that

$$\|u - v, b_2, \dots, b_n\| < \|\phi(u) - \phi(v), b_2, \dots, b_n\| = \|u - v, b_2, \dots b_n\|$$

which is a contradiction, therefore ϕ must not have more than one fixed point.

Step 2. Now, define

$$f : \mathbb{V} \to \mathbb{K}, \qquad f(x) := \|x - \phi(x), b_2, \dots, b_n\|.$$

Since the n-norm is continuous, $\mathcal{H}$ is compact, then there is a $u \in \mathcal{H}$ which

$$f(u) = \|u - \phi(u), b_2, \dots b_n\| = \inf\{f(u) : u \in \mathcal{H}\}$$

fulfills. Suppose that $\phi(u) \neq u$. Then we have

$$[f(\phi(u)) = \|\phi(u) - \phi(\phi(u)), b_2, \dots, b_n\| < \|u - \phi(u), b_2, \dots, b_n\| = f(u),$$

which would be a contradiction, so $\phi(u) = u$ and we've obtained a unique fixed point of ϕ.

3.2 The generalized Weissinger fixed point theorem

Originally, we wanted to generalize the Weissinger fixed point theorem (cf. [15], [8]). We've succeeded doing so: we have formulated and proven it. Unfortunately, our supervisors have later found it in [4] without hint of the original one. As we realized that someone has proven the theorem before us, we have read the proof rigorously and detected a mistake. In the sake of completeness we formulate the theorem correctly and give its proof, furthermore we analyze the formulation as in [4] is and then show why it is not correct.

Theorem. Let $(\mathbb{V}, \|\cdot, \dots, \cdot\|)$ be b-complete n-normed space and T be a mapping of $\mathbb{V}$ into itself. Suppose that

1. $(\alpha_n) \in l_1$ for some sequence $\alpha_n \in (0, +\infty)$ $(n \in \mathbb{N}_0)$;
2. for each integer $m \in \mathbb{N}_0$

$$\left\|T^{[m]}(u) - T^{[m]}(v), b_2, \dots, b_n\right\| \leq \alpha_m \|u - v, b_2, \dots, b_n\| \qquad (u, v \in \mathbb{V}).$$

Then $T : \mathbb{V} \to \mathbb{V}$ has a unique fixed point.

Proof. Let $x_0 \in \mathbb{V}$ arbitrary and define the sequence (x_m) as follows: $x_m := T^{[m]}x_0$. From the triangle inequality, for $p, q \in \mathbb{N}$, $p > q$ we have

$$\|x_p - x_q, b_2, \dots, b_n\| \leq \sum_{m=q}^{p-1} \|x_j - x_{j+1}, b_2, \dots, b_n\| = \sum_{m=q}^{p-1} \left\|T^{[m]}x_0 - T^{[m]}x_1, b_2, \dots, b_n\right\| \leq$$

$$\leq \sum_{m=q}^{p-1} \alpha_m \|x_0 - x_1, b_2, \dots, b_n\|$$

Let $r \in \mathbb{N}$ fulfilling $\|x_0 - x_1, b_2, \dots, b_n\| < r$, Since $(\alpha_n) \in \ell_1$, for every $\epsilon > 0$ there exists m_0 positive integer such that for every $p > q \geq m_0$

$$\sum_{m=q}^{p-1} \alpha_m < \frac{\epsilon}{r}.$$

Then it follows that

$$\|x_p - x_q, b_2, \ldots, b_n\| < \frac{\epsilon}{r} \|x_0 - x_1, b_2, \ldots, b_n\| < \epsilon,$$

which shows us that (x_m) is b-Cauchy. Since $\mathbb{V}$ is b-complete, there exists $x^* \in \mathbb{V}$ such that $\lim_{m\to\infty} x_m = x^*$.

$$\begin{aligned} \|x^* - Tx^*, b_2, \ldots, b_n\| &\leq \|x^* - x_{m+1}, b_2, \ldots, b_n\| + \|x_{m+1} - Tx^*, b_2, \ldots, b_n\| \\ &= \|x^* - x_{m+1}, b_2, \ldots, b_n\| + \|Tx_m - Tx^*, b_2, \ldots, b_n\| \\ &\leq \|x^* - x_{m+1}, b_2, \ldots, b_n\| + \alpha_1 \|x_m - x^*, b_2, \ldots, b_n\| \end{aligned}$$

Since $\lim_{m\to\infty} (x_m) = x^*$, both terms converge to zero thus $\|x^* - Tx^*, b_2, \ldots, b_n\| = 0$ meaning $Tx^* = x^*$, thereby we've proven the existence of a fixed point. As for the uniqueness, suppose x^*, y^* are fixed points of T. Then we get

$$\|x^* - y^*, b_2, \ldots, b_n\| = \left\|T^{[m]}x^* - T^{[m]}y^*, b_2, \ldots b_n\right\| \leq \alpha_m \|x^* - y^*, b_2, \ldots, b_n\|.$$

Because $\|x^* - y^*, b_2, \ldots, b_n\|$ is finite and since $(\alpha_m) \in \ell_1$ and so $\lim_{m\to\infty} (\alpha_m) = 0$ we know that $\|x^* - y^*, b_2, \ldots, b_n\| < \epsilon$ for all $\epsilon > 0$, therefore $x^* = y^*$. ■

We have to remark that in case of the 3-normed space $(\mathbb{R}^3, \|\cdot, \cdot, \cdot\|)$ if $\mathbf{b}_2 \in \mathbb{R}^3$ and $\mathbf{b}_3 \in$ are linearly dependent vectors, then from the axioms of the norm it follows that for all $\mathbf{r} \in \mathbb{R}^3$

$$\|\mathbf{r}, \mathbf{b}_2, \mathbf{b}_3\| = 0 \qquad /\text{e.g.} \quad \det[\mathbf{r}, \mathbf{b}_2, \mathbf{b}_3] = 0/$$

holds. From this, we know that every sequence is a b-Cauchy sequence and that every sequence is b-convergent, hence $\mathbb{R}^3$ is b-complete. As a consequence for every $T : \mathbb{R}^3 \to \mathbb{R}^3$ map and for every integer $m \in \mathbb{N}_0$ the condition

$$\left\|T^{[m]}\mathbf{u} - T^{[m]}\mathbf{v}, \mathbf{b}_2, \mathbf{b}_3\right\| \leq \alpha_m \|\mathbf{u} - \mathbf{v}, \mathbf{b}_2, \mathbf{b}_3\|$$

is met for every positive $(a_k) \in l_1$ sequence. However, the mappings

$$T_{\mathbf{a}} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{r} \mapsto \mathbf{r} + \mathbf{a}$$

have no fixed point for any $\mathbf{a} \neq \mathbf{0}$, since from $T_{\mathbf{a}}\mathbf{r} = \mathbf{r}$ we have $\mathbf{r} + \mathbf{a} = \mathbf{r}$ which implies $\mathbf{a} = \mathbf{0}$. This means that the independence of the vector system $b_2, \ldots, b_n$ can't be omitted in the Weissinger fixed point theorem.

3.3 Banach fixed point theorem

Lemma. Let $(\mathbb{V}, \|\cdot,\dots,\cdot\|)$ be an n-Banach space, and $\emptyset \neq A_{n+1} \subset A_n \subset \mathbb{V}$ an antitonic sequence of closed sets such that

$$\lim_{n\to\infty} (\operatorname{diam}(A_n)) = 0.$$

Then

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset.$$

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that every a_j is an arbitrary element from A_j $(j \in \mathbb{N})$. It is easy to see that this is a Cauchy-sequence, and since $\mathbb{V}$ is n-Banach space, it is also convergent. We then get a unique $\alpha \in \mathbb{V}$ with property $\alpha = \lim(a_n)$. We have to show that for every integer $n \in \mathbb{N}$ $\alpha \in A_n$ holds. Indeed, otherwise there would exist an integer $N \in \mathbb{N}$ with $\alpha \in A_N^C$, which is an open set, meaning that there exists a $U(\alpha)$ neighborhood for which $U(\alpha) \subset A_N^C$ hold, this implies

$$U(\alpha) \subset A_n^C \qquad (N \leq n \in \mathbb{N}).$$

Since $\alpha = \lim(a_n)$, there exists $M \in \mathbb{N}$ for which

$$a_n \in U(\alpha) \qquad (M \leq n \in \mathbb{N})$$

So, for $n > \max\{M,N\}$ we have $a_n \in A_n \cap A_n^C$, which is a contradiction. This has clearly a consequence that

$$\alpha \in \bigcap_{n\in\mathbb{N}} A_n \qquad \text{implies} \qquad \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset. \quad \blacksquare$$

Theorem. Let $(\mathbb{V}, \|\cdot,\dots,\cdot\|)$ be an n-Banach space, $\phi : \mathbb{V} \to \mathbb{V}$ a b-contraction, i.e. there exists $q \in [0,1)$ such that

$$\|\phi(u) - \phi(v), b_2, \dots, b_n\| \leq q \|u - v, b_2, \dots, b_n\| \qquad (u,v \in \mathbb{V}).$$

Then ϕ has a unique fixed point.

Proof.

Step 1. We show that the mapping

$$f : \mathbb{V} \to \mathbb{V}, \qquad f(x) := \|x - \phi(x), b_2, \dots, b_n\|$$

is uniform continuous. For every $x,y \in \mathbb{V}$ we have

$$\begin{aligned} f(x) - f(y) &= \|x - \phi(x), b_2, \ldots, b_n\| - \|y - \phi(y), b_2, \ldots, b_n\| \\ &\leq \|x - y, b_2, \ldots b_n\| + \|y - \phi(y), b_2, \ldots b_n\| \\ &\quad + \|\phi(y) - \phi(x), b_2, \ldots, b_n\| - \|y - \phi(y), b_2, \ldots, b_n\| \\ &= \|x - y, b_2, \ldots, b_n\| + \|\phi(y) - \phi(x), b_2, \ldots, b_n\| \\ &\leq (1+q)\|x - y, b_2, \ldots, b_n\| \end{aligned}$$

Similarly,

$$f(y) - f(x) \leq (1+q)\|x - y, b_2, \ldots, b_n\|.$$

This means that for every $x,y \in \mathbb{V}$

$$\|f(x) - f(y)\| \leq (1+q)\|x - y, b_2, \ldots, b_n\|$$

holds, from which the uniform continuity of f follows.

Step 2. As a consequence the set

$$A_n := \left\{ x \in \mathbb{V} : f(x) \leq \frac{1}{n} \right\}$$

is closed for every integer $n \in \mathbb{N}$. It is also clear that the sequence $(A_n)_{n\in\mathbb{N}})$ is antitone, furthermore for every $x \in \mathbb{V}$ we have

$$\begin{aligned} f\left(\phi^{[n]}(x)\right) &= \left\|\phi^{[n]}(x) - \phi^{[n+1]}(x), b_2, \ldots b_n\right\| \\ &\leq q\left\|\phi^{[n-1]}(x) - \phi^{[n]}(x), b_2, \ldots b_n\right\| \leq q^n \|x - \phi(x), b_2, \ldots b_n\|. \end{aligned}$$

Because q^n tends to zero as $n \to \infty$, therefore for every $x \in \mathbb{V}$ we have $\phi^{[n]}(x) \longrightarrow 0 \in \mathbb{V}$ $(n \to \infty)$. This implies that for all $n \in \mathbb{N}$, A_n is a nonempty set.

Step 3. From the triangle inequality we have that $n \in \mathbb{N}$ and for all $x,y \in A_n$ the estimation

$$\begin{aligned} \|x - y, b_2, \ldots, b_n\| &\leq \|x - \phi(x), b_2, \ldots, b_n\| + \|\phi(x) - \phi(y), b_2, \ldots, b_n\| \\ &\quad + \|\phi(y) - y, b_2, \ldots, b_n\| \leq f(x) + q\|x - y, b_2, \ldots, b_n\| + f(y) \\ &= \frac{1}{n} + q\|x - y, b_2, \ldots, b_n\| + \frac{1}{n} \end{aligned}$$

which means that

$$\|x - y, b_2, \ldots b_n\| \leq \frac{2}{n(1-q)},$$

i.e. $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$. Since $\mathbb{V}$ is an n-Banach space,

$$\mathcal{A} := \bigcap_{n=1}^{\infty} A_n \neq \emptyset.$$

If $\alpha,\beta \in \mathcal{A}$, $\alpha \neq \beta$, we know that

$$\|\alpha - \beta, b_2, \ldots b_n\| \leq \operatorname{diam}(\mathcal{A}) = \lim(\operatorname{diam}(A_n)) = 0.$$

Therefore $\alpha = \beta$, so there is a unique $\alpha \in \mathcal{A}$ for which $f(\alpha) = 0$, i.e. $\phi(\alpha) = \alpha$. ■

Chapter 4

$\mathfrak{n}$-Hilbert spaces

In what follow we've generalized three very important theorems from functional analysis.

4.1 Linear Lax-Milgram theorem

Theorem. Let T be a b-bounded, b-elliptic, b-sesquilinear form defined on

$$\mathbb{H} \times \mathbb{H} \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle,$$

for which there exist $k,K \in \mathbb{R}$, $0 < k \leq K < \infty$ where the following hold:

1. $|T(u,v,b_2,\dots b_n)| \leq K \|u,b_2,\dots,b_n\| \, \|v,b_2,\dots,b_n\|$;
2. $|T(u,u,b_2,\dots,b_n)| \geq k \|u,b_2,\dots,b_n\|^2$.

Then there exists a unique bijective $A \in L(\mathbb{H},\mathbb{H})$, for which $\|A\| \leq K$ and for every $u,v \in \mathbb{H}$

$$T(u,v,b_2,\dots,b_n) = \langle Au,v|b_2,\dots,b_n\rangle.$$

It also holds true that $A^{-1} \in L(\mathbb{H},\mathbb{H})$ and that $\left\|A^{-1}\right\| \leq \frac{1}{k}$.

Proof.

Step 1. Define

$$\Phi_v : \mathbb{H} \times \langle b_2 \rangle \times \dots \langle b_n \rangle \to \mathbb{K}, \quad \Phi_v(u,b_2,\dots,b_n) := T(u,v,b_2,\dots,b_n).$$

Since T is b-bounded,

$$\begin{aligned} |\Phi_v(u)| &= |T(u,v,b_2,\dots,b_n)| \leq K \|u,b_2,\dots,b_n\| \, \|v,b_2,\dots,b_n\| \\ &= K_b \|v,b_2,\dots,b_n\|, \end{aligned}$$

hence $\Phi_v \in T^*$ and

$$\|\Phi_v\| = \|T(\cdot, v, b_2, \ldots, b_n)\| \leq K \|v, b_2, \ldots, b_n\|.$$

So from the generalized Riesz representation theorem there exists a unique $a \in \mathbb{H}$, $a, b_2, \ldots, b_n$ linearly independent, such that

$$\Phi_v(u, b_2, \ldots, b_n) = \langle u, a | b_2, \ldots b_n \rangle \qquad \text{and} \qquad \|\Phi_v\| = \|a, b_2, \ldots, b_n\|.$$

Step 2. Since T is b-sesquilinear,

$$C : \mathbb{H} \to \mathbb{H},\ C(v) = a$$

operator is linear, since for all $u, v_1, v_2 \in \mathbb{H}$, and for all $\lambda \in \mathbb{K}$

$$\begin{aligned}
\langle u, C(v_1 + v_2) | b_2, \ldots, b_n &= T(u, a_1 + a_2, b_2, \ldots, b_n) \\
&= T(u, a_1, b_2, \ldots, ,b_n) + T(u, a_2, b_2, \ldots, b_n) \\
&= \langle u, Cv_1 | b_2, \ldots, b_n \rangle + \langle u, Cv_2 | b_2, \ldots, b_n \rangle \\
\langle u, C(\lambda v) | b_2, \ldots, b_n \rangle &= T(u, \lambda a, b_2, \ldots, b_n) = \bar{\lambda} T(u, a, b_2, \ldots, b_n) \\
&= \bar{\lambda} \langle u, Cv, b_2, \ldots, b_n \rangle = \langle u, \lambda Cv, b_2, \ldots, b_n \rangle
\end{aligned}$$

Additionally,

$$\|Cv\| = \|a\| = \|\Phi_v\| = \|T(\cdot, v, b_2, \ldots, b_n)\| \leq K \|v, b_2, \ldots, b_n\|,$$

thus we've obtained that $C \in L(\mathbb{H}, \mathbb{H})$, $\|C\| \leq K$. Thus, for $A = C^*$:

$$T(u, v, b_2, \ldots, b_n) = \langle Au, v | b_2, \ldots, b_n \rangle \quad \forall\, u, v \in \mathbb{H}$$

Step 3. Since T is b-elliptic,

$$\begin{aligned}
k \|u, b_2, \ldots, b_n\|^2 &\leq \Re(T(u, u, b_2, \ldots, b_n)) = \Re(|\langle Au, u | b_2, \ldots, b_n \rangle|) \\
&\leq |\langle Au, u | b, \ldots, b_n \rangle| \leq \|Au, b_2 \ldots, b_n\| \, \|u, b_2, \ldots, b_n\|
\end{aligned}$$

Step 4. Thus, for every

$$u \in \mathbb{H} : k \|u, b_2, \ldots, b_n\| \leq \|Au, b_2, \ldots, ,b_n\|,$$

meaning that $R(A)$ is a closed subspace, A is injective and $\left\|A^{-1}\right\| \leq \frac{1}{k}$.

Step 5. Suppose $R(A) \neq \mathbb{H} \times \langle b_2 \rangle \times \ldots \langle b_n \rangle$ From closedness we know that there exists $\xi \in R(A)^{\perp} \setminus \{0\}$, and let $a := A\xi$.

$$k \|\xi, b_2, \ldots, b_n\| \leq \Re(T(\xi, \xi, b_2, \ldots, b_n)) = \Re(\langle A\xi, \xi | b_2, \ldots, b_n \rangle) = 0$$

implying $\xi = 0$, which is a contradiction, thus A is bijective and the proof is complete. ■

4.2 Zarantonello theorem

Theorem. Let $(\mathbb{H},\langle\cdot,\cdot\,|\,\cdot,\ldots,\cdot\rangle$ be a real n-Hilbert space with $\|\cdot,\ldots,\cdot\| = \sqrt{\langle\cdot,\ldots,\cdot\rangle}$ and suppose for $A : \mathbb{H} \to \mathbb{H}$ operator there exists $0 < \alpha < 1$ and $L > 0$ where for every $x,y \in \mathbb{H}$ the following hold:

1. $\langle Ax - Ay, x - y | b_2,\ldots,b_n\rangle \geq \alpha \|x - y, b_2,\ldots,b_n\|$;
2. $\|Ax - Ay, b_2,\ldots,b_n\| \leq L \|x - y, b_2,\ldots b_n\|$.

Then, for every $v \in \mathbb{H}$ there exists a unique $u \in \mathbb{H}$, for which $Au = v$.

Proof. There exists a unique $u \in \mathbb{H} : Au = v$ if and only if for some $\epsilon > 0$ u is a fixed point of

$$\Phi_v : \mathbb{H} \to \mathbb{H}, \qquad \Phi_v(x) = x - \epsilon\,(Ax - v)\,.$$

For every $x,y \in \mathbb{H}$ we have

$$\begin{aligned}
\|\Phi_v(x) - \Phi_v(y), b_2,\ldots,b_n\|^2 &= \|x - y, b_2,\ldots,b_n\|^2 - 2\epsilon\langle Ax - Ay, x - y | b_2,\ldots,b_n\rangle \\
&\quad + \epsilon^2 \|Ax - Ay, b_2,\ldots b_n\| \\
&\leq \|x - y, b_2,\ldots,b_n\|^2 - 2\epsilon\alpha \|x - y, b_2,\ldots,b_n\|^2 \\
&\quad + \epsilon^2 L^2 \|x - y, b_2,\ldots,b_n\| \\
&= \left(1 - 2\epsilon\alpha + \epsilon^2 L^2\right) \|x - y, b_2,\ldots b_n\|\,.
\end{aligned}$$

Observe, that

$$k := 1 - 2\epsilon\alpha + \epsilon^2 L^2 \geq 0,$$

and by choosing $\epsilon := \frac{2\alpha}{L^2}$ we get that $k = 1$. Then for all $\epsilon \in \mathbb{R}$, $0 < \epsilon < \frac{2\alpha}{L^2}$ the number $q := \sqrt{k}$ is from the interval $[0,1)$ and

$$\|\Phi_v(x) - \Phi_v(y), b_2,\ldots,b_n\| \leq q \|x - y, b_2,\ldots b_n\|\,,$$

hence for all $v \in \mathbb{H}$ Φ_v is a b-contraction. From the generalized Banach fixed-point theorem it follows that for every $v \in \mathbb{H}$ there exists a unique $u \in \mathbb{H}$ for which

$$\Phi_v(u) = u = u - \epsilon\,(Au - v)$$

if and only if $Au = v$ ■

4.3 The nonlinear Lax-Milgram theorem

Theorem. Let $(\mathbb{H},\langle\cdot,\dots,\cdot\rangle)$ be a real n-Hilbert space with $\|\cdot,\dots,\cdot\| = \sqrt{\langle\cdot,\dots,\cdot\rangle}$ and let $T : \mathbb{H}\times\mathbb{H}\times\langle b_2\rangle\times\cdots\times\langle b_n\rangle \to \mathbb{R}$ and $S : \mathbb{H}\times\langle b_2\rangle\times\cdots\times\langle b_n\rangle \to \mathbb{R}$ linear functionals, satisfying the following:

1. for every $\alpha\in\mathbb{H}$ the functional

$$f_\alpha : \quad \mathbb{H}\to\mathbb{R}, \quad f_\alpha := T(\alpha,x,b_2,\dots,b_n)$$

   is b-continous and linear;

2. for some $\gamma > 0$ and for every $x,y\in\mathbb{H}$

$$\gamma\|x-y,b_2,\dots,b_n\|^2 \le T(x,x-y,b_2,\dots,b_n) - T(y,x-y,b_2,\dots,b_n)$$

   holds;

3. there exists an $L > 0$ such that for every $x,y,z\in\mathbb{H}$

$$|T(x,z,b_2,\dots,b_n) - T(y,z,b_2,\dots,b_n)| \le L\|x-y,b_2,\dots b_n\|\,\|z,b_2,\dots,b_n\|$$

   holds;

4. S is linear and continous.

Then for all $v\in\mathbb{H}$ there is a unique $u\in\mathbb{H}$ fulfilling $T(v,u,b_2,\dots,b_n) = S(u,b_2,\dots,b_n)$.

Proof. From the generalized Riesz-theorem we know that for every $z\in\mathbb{H}$ there is a unique $R\in L(\mathbb{H},\mathbb{H})$, for which

$$T(z,y,b_2,\dots,b_n) = \langle Rz,y|b_2,\dots,b_n\rangle$$

for every $y\in\mathbb{H}$, hence from the second and third conditions we have

1. $\gamma\|x-y,b_2,\dots,b_n\|^2 \le \langle Rx-Ry,x-y|b_2,\dots,b_n\rangle$;
2. $|\langle Rx-Rx,z|b_2,\dots b_n\rangle| \le L\|x-y\|\ \forall\, x,y\in\mathbb{H}$.

From these we know that

$$\|Rx-Ry,b_2,\dots b_n\| = \sup\{\langle Rx-Ry,\alpha|b_2,\dots,b_n\rangle : \alpha\in\mathbb{H}, \|\alpha,b_2,\dots,b_n\|$$

$$\le L\|x-y,b_2,\dots,b_n\|.$$

Also, by using the generalized Riesz-theorem again we get that there exists a unique $v\in\mathbb{H}$ for which $S(u,b_2,\dots,b_n) = \langle v,u|b_2,\dots,b_n\rangle$ for all $u\in\mathbb{H}$. From this it follows that $T(u,v,b_2,\dots,b_n) = S(u,b_2,\dots b_n)$ if and only if $Ru = v$, which has a unique solution from the generalized Zarantonello theorem. ■

Bibliography

[1] Brzdęk, J. ; Ciepliński, K.: *A fixed point theorem in n-Banach spaces and Ulam stability*, Journal of Mathematical Analysis and Applications, **470**(1) (2019), 632–646.

[2] Diminnie, C.; Gähler, S.; White, A.: *2-inner product spaces II*, Demonstratio Mathematica, **10**(1) (1977), 169–188.

[3] Gähler, S.: *Lineare 2-normierte Ráume*, Math. Nachr., **28** (1964), 1–43.

[4] Ghosh, P.; Samanta, T. K.: *A few fixed point theorems in linear n-normed space*, arXiv preprint arXiv:2210.07849.

[5] Ghosh, P.; Samanta, T. K.: *Generalized Riesz Representation Theorem in* n*-Hilbert space*, arXiv preprint arXiv:2203.04293.

[6] Ghosh, P.; Samanta, T. K.: *Reflexivity of linear* n*-normed space with respect to b-linear functional*, arXiv preprint arXiv:2101.09661.

[7] Ghosh, P.; Roy, S.; Samanta, T. K.: *Uniform Boundedness Principle and Hahn-Banach Theorem for b-linear functional related to linear 2-normed space*, arXiv preprint arXiv:2101.00653.

[8] Kovács, S.: *Funkcionálanalízis feladatokban, egyetemi jegyzet*, Budapest, 2013.
ISBN: 978-963-284-445-9
(http://numanal.inf.elte.hu/~alex/hu/anyag/PROGINF/FunkAnal/FunkAnalKS.pdf)

[9] Kovács, S.: *Analízis IV*, Budapest, 2021.
(https://numanal.inf.elte.hu/~alex/hu/anyag/anal4/kiadvanyok/Analizis4GyakorlatKS.pdf)

[10] Kreyszig, E.: *Introductory functional analysis with applications*, John Wiley & Sons, New York-London-Sydney, 1978.

[11] Simon, P.: *A funkcionálanalízis alapjai*, ELTE Eötvös Kiadó, 2017.

[12] Misiak, A.: n*-inner product spaces*, Math. Nachr., **140**(1) (1989), 299–319.

[13] Reddy, G. U.: *Some fixed point results in 2-normed spaces*, Nonlinear Studies, **30**(2) (2023), 387–391.

[14] SCHIPP, F.: *Analízis III*, Budapest, 2003.
(https://numanal.inf.elte.hu/~schipp/Jegyzetek/Anal_3.pdf)

[15] WEISSINGER, J.: *Zur Theorie und Anwendung des Iterationsverfahrens*, Math. Nachr. **8** (1952), 193–212.